

# SPIN-PARTICLE CONNECTIONS

*C.D. Batista and G. Ortiz*

Theoretical Division, Los Alamos National Laboratory  
Los Alamos, NM 87545, USA

## 1. INTRODUCTION

Group theory and geometry have been fundamental to the physics of the Twentieth Century. The notion of symmetry has shaped our current conception of nature; however, nature is also full of symmetry breakings. Therefore, understanding the idea of invariance and its corresponding conservation laws is as fundamental as determining the causes that prevent such harmony and leads to more complex behavior. These concepts are at the heart of the field of quantum phase transitions that studies the changes that can occur in the macroscopic properties of matter at zero temperature due to changes in the parameters characterizing the system.

On the other hand, the notion of algebra and its homomorphisms have also been essential to unravel hidden structures in theoretical physics: Internal symmetries which are hidden in one algebraic representation of a model become manifest in the other one. In 1928 Jordan and Wigner [1] made a first step relating quantum spin  $S = \frac{1}{2}$  degrees of freedom to particles with fermion statistics, an application being the mapping of the isotropic XY model describing planar magnets onto a tight-binding spinless fermion model which can be exactly solved in one spatial dimension. From the group theoretical viewpoint, an internal  $U(1)$  continuous symmetry (related to particle number conservation), for instance, is evidenced in the fermion representation of the XY model which was hidden in the spin representation. Overall, what Jordan and Wigner established was an isomorphism of  $*$ -algebras, i.e., an isomorphism between the Pauli and fermion algebras.

The three basis elements  $S_j^\mu$  ( $\mu = x, y, z$ ) (linear and Hermitian operators) of the Lie algebra  $su(2)$  for each lattice site  $j$  ( $j = 1, \dots, L$ ) satisfy the equal-time commutation relations [2]

$$[S_j^\mu, S_k^\nu] = i\delta_{jk}\epsilon_{\mu\nu\lambda}S_j^\lambda, \quad (1)$$

with  $\epsilon$  the totally antisymmetric Levi-Civita symbol. Equivalently, in terms of the ladder operators  $S_j^\pm = S_j^x \pm iS_j^y$

$$[S_j^+, S_j^-] = 2S_j^z, \quad [S_j^z, S_j^\pm] = \pm S_j^\pm, \quad \{S_j^+, S_j^-\} = 2(S(S+1) - (S_j^z)^2), \quad (2)$$

where  $2S + 1$  is the dimension of the irreducible representation  $S$ . Mathematically, the Jordan-Wigner transformation [1] involves the  $S=\frac{1}{2}$  irreducible representation of the Lie group  $SU(2)$ . The mapping itself establishes the isomorphism of algebras:  $S_j^\mu = \mathcal{F}(\{c_m\}, \{c_m^\dagger\})$ , where  $\mathcal{F}$  is an operator function of the “spinless” fermions  $(c_m, c_m^\dagger)$ , for different modes  $m$  ( $m = 1, \dots, L$ ), satisfying canonical anticommutation relations

$$\{c_j, c_m\} = \{c_j^\dagger, c_m^\dagger\} = 0, \quad \{c_j, c_m^\dagger\} = \delta_{jm}. \quad (3)$$

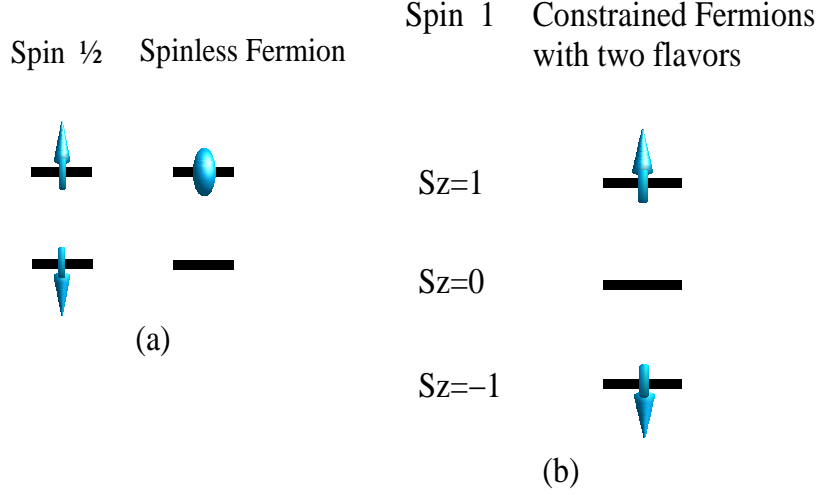
Up until now, no one was able to generalize Jordan and Wigner’s findings for arbitrary spin  $S$ , spatial dimension and particle statistics. In the present manuscript we generalize their spin-fermion mapping to any irreducible representation  $S$ . Our mappings are valid for regular lattices in any spatial dimension  $d$  and particle statistics. The significance of these transformations is that they help us understand various aspects of the same physical system by transforming intricate interaction terms in one representation into simpler ones in the other. Problems which seem untractable can even be exactly solved after the mapping. In other cases, new and better approximations can, in principle, be realized since fundamental symmetries which are hidden in one representation are manifest in the other.

From a physical viewpoint what our spin-particle transformations achieve is an exact connection between models of localized quantum spins  $S$  to models of itinerant particles with  $(2S = N_f)$  color degrees of freedom or “effective” spin  $s = S - \frac{1}{2}$ . As we will see this dictionary turns out to be extremely useful to understanding the various complex quantum phases that arise in models of itinerant strongly correlated systems, as a result of the competing interactions. In the next Section, we start by analyzing the one-dimensional  $S=1$  case. Then, we will show a generalization to arbitrary spin, spatial dimension and, finally, particle statistics (i.e., general spin-anyon mappings for the case  $d \leq 2$ ). We illustrate the power of these transformations by showing exact solutions to  $1d$  lattice models previously unsolved by standard techniques. We also present a proof of the existence of the Haldane gap in  $S=1$  bilinear nearest-neighbors Heisenberg spin chains and discuss the relevance of the mapping to models of strongly correlated electrons.

## 2. $S = 1$ MAPPING

To gain some intuition about the mapping between particles and spins, it is instructive to start by analyzing the local Hilbert spaces associated with each world, i.e., the Hilbert spaces associated with a single site (mode) in each representation. In the traditional Jordan-Wigner transformation [1], a spin  $S = \frac{1}{2}$  is mapped onto a spinless fermion. Both local Hilbert spaces are shown in Fig. 1 (a), and from that figure it is clear that one can make a one-to-one mapping between the states of both spaces. The convention used by Jordan and Wigner was to map the up (down) state into the occupied (empty) state. The next step consists of mapping the corresponding operators acting on one site.  $S_j^z$  and  $n_j = c_j^\dagger c_j$  are diagonal operators in each representation and they are related through the following expression

$$S_j^z = n_j - \frac{1}{2}. \quad (4)$$



**Figure 1.** Local Hilbert spaces for the spin and fermion worlds at site (mode)  $j$ : (a)  $S = \frac{1}{2}$ , (b)  $S = 1$ .

$S_j^+$  and  $S_j^-$  are non-diagonal in the present basis and must be related to  $c_j^\dagger$  and  $c_j$ . If we consider the problem on a lattice, the fermions can be permuted; however, the commutation relations of the two representations are different. To take into account the transmutation of statistics, one must introduce a non-local operator  $K_j$ . Then, the transformation for the non-diagonal operators is [1]

$$S_j^+ = c_j^\dagger K_j, \quad S_j^- = K_j^\dagger c_j, \quad (5)$$

where  $K_j = \exp[i\pi \sum_{k < j} n_k]$ . This operator introduces a negative sign each time one fermion hops over the fermion at site  $j$ . This negative sign cancels the sign coming from the fermionic statistics. The result is the transmutation of fermionic anticommutation relations into spin commutation relations. Note that  $K_j$  does not appear in the expression for  $S_j^z$  since  $n_j$  is bilinear in the fermion operators.

This idea can be generalized to  $S = 1$  following similar steps to the ones described above. Figure 1 (b) shows the two local Hilbert spaces. In this case, the particle Hilbert space corresponds to spin- $\frac{1}{2}$  (two flavors) fermions with the constraint of no double occupancy. This constraint can be taken into account by introducing the Hubbard operators  $\bar{c}_{j\sigma}^\dagger = c_{j\sigma}^\dagger (1 - n_{j\bar{\sigma}})$  and  $\bar{c}_{j\sigma} = (1 - n_{j\bar{\sigma}}) c_{j\sigma}$  ( $\sigma = 1, -1$ ;  $\bar{\sigma} = -\sigma$ ), which form a subalgebra of the so-called double graded algebra  $Sp(1, 2)$  [3]. From Fig. 1 (b) one realizes that  $S_j^z$  is the difference between the occupation numbers of the two different fermion flavors.  $S_j^+$  must be a linear combination of annihilation and creation operators since we need to annihilate (see Fig. 1 (b)) one fermion to go from  $S_j^z = -1$  to the  $S_j^z = 0$  state and to create the other fermion to go from  $S_j^z = 0$  to  $S_j^z = 1$ . To simplify notation we introduce the following composite operators

$$f_j^\dagger = \bar{c}_{j1}^\dagger + \bar{c}_{j\bar{1}}^\dagger, \quad f_j = \bar{c}_{j1} + \bar{c}_{j\bar{1}}. \quad (6)$$

For spins on a lattice we again fermionize the spins and reproduce the correct spin algebra with the following transformation

$$S_j^+ = \sqrt{2} (\bar{c}_{j1}^\dagger K_j + K_j^\dagger \bar{c}_{j\bar{1}}), \quad S_j^- = \sqrt{2} (K_j^\dagger \bar{c}_{j1} + \bar{c}_{j\bar{1}}^\dagger K_j), \quad S_j^z = \bar{n}_{j1} - \bar{n}_{j\bar{1}}, \quad (7)$$

whose inverse manifests the nonlocal character of the mapping

$$f_j^\dagger = \frac{1}{\sqrt{2}} \exp[i\pi \sum_{k<j} (S_k^z)^2] S_j^+ , \quad f_j = \frac{1}{\sqrt{2}} \exp[-i\pi \sum_{k<j} (S_k^z)^2] S_j^- ,$$

$$\bar{c}_{j1}^\dagger = S_j^z f_j^\dagger , \quad \bar{c}_{j1} = f_j S_j^z , \quad \bar{c}_{j\bar{1}}^\dagger = -S_j^z f_j , \quad \bar{c}_{j\bar{1}} = -f_j^\dagger S_j^z , \quad (8)$$

where the string operators  $K_j = \exp[i\pi \sum_{k<j} \bar{n}_k] = \prod_{k<j} \prod_\sigma (1 - 2\bar{n}_{k\sigma})$  are the natural generalizations of the ones introduced before [4]. The number operators are defined as  $\bar{n}_k = \bar{n}_{k1} + \bar{n}_{k\bar{1}}$  ( $\bar{n}_{k\sigma} = \bar{c}_{k\sigma}^\dagger \bar{c}_{k\sigma}$ ). These  $f$ -operators have the remarkable property that

$$\{f_j^\dagger, f_j\} = \{S_j^+, S_j^-\} , \quad (9)$$

which suggests an analogy between spin operators and “constrained” fermions.

### 3. $S = 1$ MODELS

#### 3.1 Haldane Systems

Generically, half-odd integer spin chains have a qualitatively different excitation spectrum than integer spin chains. The Lieb, Schultz, Mattis and Affleck theorem [5] establishes that the half-odd integer antiferromagnetic (AF) bilinear nearest-neighbors (NN) Heisenberg chain is gapless if the ground state is non-degenerate. The same model with integer spins is conjectured to have a Haldane gap [6]. To understand the origin of the Haldane gap we analyze the form of the  $1d$   $S=1$  XXZ Hamiltonian using the above representation (an overall omitted constant  $J > 0$  determines the energy scale)

$$H_{\text{xxz}} = \sum_j S_j^z S_{j+1}^z + \Delta (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) = \sum_j H_j^z + H_j^{\text{xx}} . \quad (10)$$

It is easy to show that the constrained fermion version of this Hamiltonian is a ( $S = \frac{1}{2}$ )  $t$ - $J_z$  model [7] plus particle non-conserving terms which break the  $U(1)$  symmetry

$$H_{\text{xxz}} = \sum_j (\bar{n}_{j1} - \bar{n}_{j\bar{1}})(\bar{n}_{j+11} - \bar{n}_{j+1\bar{1}}) + \Delta \sum_{j\sigma} (\bar{c}_{j\sigma}^\dagger \bar{c}_{j+1\sigma} + \bar{c}_{j\sigma}^\dagger \bar{c}_{j+1\bar{\sigma}} + \text{H.c.}) . \quad (11)$$

The charge spectrum of the ( $S = \frac{1}{2}$ )  $t$ - $J_z$  model is gapless but the spin spectrum is gapped due to the explicitly broken  $SU(2)$  symmetry (Luther-Emery liquid) [7]. The proof of the existence of this spin gap is given in Appendix I. Therefore, the spectrum of the  $S=1$  Hamiltonian associated with the  $t$ - $J_z$  model, with  $t = -\Delta$  and  $J_z = 4$ , (which has only spin excitations) is gapless. Hence the term which explicitly breaks  $U(1)$  must be responsible for the opening of the Haldane gap. We can prove this by considering the perturbative effect that the interaction  $\eta \sum_{j\sigma} (\bar{c}_{j\sigma}^\dagger \bar{c}_{j+1\bar{\sigma}} + \text{H.c.})$  has on the  $t$ - $J_z$  Hamiltonian. To linear order in  $\eta$  ( $> 0$ ), Eq. (11) maps onto the ( $S = \frac{1}{2}$ ) XYZ model with  $\mathcal{J}_x = 2(\eta + \Delta)$ ,  $\mathcal{J}_y = -2(\eta - \Delta)$ , and  $\mathcal{J}_z = -1$ . To prove

this statement we need to explain first how the lowest energy subspace of the  $t$ - $J_z$  Hamiltonian can be mapped onto a spinless  $t$ - $V$  model (the complete demonstration is presented in Ref. [7]).

The  $t$ - $J_z$  Hamiltonian represents a hole-doped Ising model

$$H_{t-J_z} = \hat{H}_{J_z} + \hat{T} = J_z \sum_j S_j^z S_{j+1}^z - t \sum_{j\sigma} \left( \bar{c}_{j\sigma}^\dagger \bar{c}_{j+1\sigma} + \text{H.c.} \right) . \quad (12)$$

Consider the set of parent states with  $M$  holes and  $L - M = N_\uparrow + N_\downarrow$  quantum particles ( $\sigma = \uparrow, \downarrow$ ),  $|\Phi_0(N_{\uparrow\uparrow}, N_{\uparrow\downarrow})\rangle$ , defined as

$$|\Phi_0(N_{\uparrow\uparrow}, N_{\uparrow\downarrow})\rangle = \underbrace{|\uparrow\uparrow\downarrow\uparrow\downarrow \cdots\rangle}_{L-M} \underbrace{|\circ\circ\circ\circ\circ\cdots\rangle}_M , \quad (13)$$

where  $N_{\uparrow\uparrow}$  ( $N_{\uparrow\downarrow}$ ) is the number of ferro (antiferro)-magnetic links ( $N_{\uparrow\uparrow} + N_{\uparrow\downarrow} = L - M - 1$ ). These states are eigenstates of the magnetic part of  $H_{t-J_z}$ ,  $\hat{H}_{J_z}$ , with energy  $E_M(N_{\uparrow\uparrow}, N_{\uparrow\downarrow}) = J_z(N_{\uparrow\uparrow} - N_{\uparrow\downarrow})/4$ , and  $z$ -component of the total spin  $(N_\uparrow - N_\downarrow)/2$ .

From a given parent state one can generate a subspace of the Hilbert space,  $\mathcal{M}(N_\uparrow, N_{\uparrow\uparrow}, N_{\uparrow\downarrow})$ , by applying the hopping operators  $\hat{T}_{j,\sigma} = c_{j\sigma}^\dagger c_{j+1\sigma} + \text{H.c.}$  ( $j = 1, \dots, L - 1$ ) to the parent state and its descendants

$$|\Phi_1(N_{\uparrow\uparrow}, N_{\uparrow\downarrow})\rangle = \hat{T}_{L-M,\sigma} |\Phi_0(N_{\uparrow\uparrow}, N_{\uparrow\downarrow})\rangle \quad (14)$$

or, in general,

$$|\Phi_\alpha(N_{\uparrow\uparrow}, N_{\uparrow\downarrow})\rangle = \hat{T}_{j,\sigma} |\Phi_\beta(N_{\uparrow\uparrow}, N_{\uparrow\downarrow})\rangle . \quad (15)$$

The dimension  $\mathcal{D}$  of the subspace  $\mathcal{M}(N_\uparrow, N_{\uparrow\uparrow}, N_{\uparrow\downarrow})$  is  $\binom{L}{M}$ . Moreover, these different subspaces are orthogonal and not mixed by the Hamiltonian  $H_{t-J_z}$ .

We want to show now that, for a given number of holes  $M$ , the subspace generated by the Néel parent state,  $\mathcal{M}(N_\uparrow, 0, N_{\uparrow\downarrow}) \equiv \mathcal{M}_0$ , contains the ground state. To this end, one has to note that the matrix elements  $\langle \Phi_\alpha(N_{\uparrow\uparrow}, N_{\uparrow\downarrow}) | \hat{T} | \Phi_\beta(N_{\uparrow\uparrow}, N_{\uparrow\downarrow}) \rangle$  are the same for the different subspaces  $\mathcal{M}$ . Nonetheless, the magnetic matrix elements  $\langle \Phi_\alpha(N_{\uparrow\uparrow}, N_{\uparrow\downarrow}) | \hat{H}_{J_z} | \Phi_\beta(N_{\uparrow\uparrow}, N_{\uparrow\downarrow}) \rangle = \delta_{\alpha\beta} A(\mathcal{N}_{\uparrow\uparrow}, \mathcal{N}_{\uparrow\downarrow})$  are different for the different subspaces, with  $A(0, \mathcal{N}_{\uparrow\downarrow}) \leq A(\mathcal{N}_{\uparrow\uparrow}, \mathcal{N}_{\uparrow\downarrow})$ ,  $\mathcal{N}_{\uparrow\downarrow} = \mathcal{N}_{\uparrow\uparrow} + \mathcal{N}_{\uparrow\downarrow}$ . [Notice that, for a generic state of a given subspace,  $\mathcal{N}_{\uparrow\uparrow} + \mathcal{N}_{\uparrow\downarrow} \leq L - M - 1$  with the equality satisfied by the parent state only, where  $\mathcal{N}_{\uparrow\uparrow} = N_{\uparrow\uparrow}$  and  $\mathcal{N}_{\uparrow\downarrow} = N_{\uparrow\downarrow}$ .] Therefore, the Hamiltonian matrices  $\mathbf{H}_{\alpha,\beta}^{\mathcal{M}}$  (of dimension  $\mathcal{D} \times \mathcal{D}$ ) in each subspace  $\mathcal{M}$ , consists of identical off-diagonal matrix elements ( $\mathbf{H}_{\alpha,\beta}^{\mathcal{M}} = \mathbf{H}_{\alpha,\beta}^{\mathcal{M}'}$ ,  $\alpha \neq \beta$ ) and different diagonal ones. These Hermitian matrices can be ordered according to the increasing value of the energy  $E_M$  of their parent states, which is equivalent (for fixed  $L$  and  $M$ ) to ordering by the increasing number of ferromagnetic links  $N_{\uparrow\uparrow}$ ,  $\mathbf{H}^{\mathcal{M}} \equiv \mathbf{H}^{N_{\uparrow\uparrow}}$ . For any  $N_{\uparrow\uparrow} < N'_{\uparrow\uparrow}$ ,  $\mathbf{H}^{N'_{\uparrow\uparrow}} = \mathbf{H}^{N_{\uparrow\uparrow}} + B$ , where  $B$  is a positive semidefinite matrix. Then, the monotonicity theorem [8] tells us that

$$E_k(N_{\uparrow\uparrow}) \leq E_k(N'_{\uparrow\uparrow}) \quad \forall k = 1, \dots, \mathcal{D} , \quad (16)$$

where  $E_k(N_{\uparrow\uparrow})$  are the eigenvalues of  $\mathbf{H}^{N_{\uparrow\uparrow}}$  arranged in increasing order. Therefore, we conclude that the lowest eigenvalue of  $H_{t-J_z}$  must be in  $\mathcal{M}_0$  and is  $E_1(0)$ .

The next step consists in showing that, within the ground state subspace  $\mathcal{M}_0$ , the Hamiltonian  $H_{t-J_z}$  maps into an attractive spinless fermion model. If one makes the following identification

$$\underbrace{|\uparrow\downarrow\uparrow\downarrow\cdots\rangle}_{L-M} \underbrace{|\circ\circ\circ\circ\cdots\rangle}_M \rightarrow \underbrace{|\bullet\bullet\bullet\cdots\rangle}_{L-M} \underbrace{|\circ\circ\circ\circ\cdots\rangle}_M, \quad (17)$$

i.e., any spin particle ( $c_{j\sigma}^\dagger$ , independently of the value of  $\sigma$ ) maps into a single spinless fermion ( $b_j^\dagger$ ) in  $\mathcal{M}_0$ , it is straightforward to realize that all matrix elements of  $H^0$  are identical to the matrix elements of

$$H_0 = -t \sum_j (b_j^\dagger b_{j+1} + \text{H.c.}) - \frac{J_z}{4} \sum_j n_j n_{j+1} \quad (18)$$

in the corresponding new basis. In Eq. (18)  $n_j = b_j^\dagger b_j$ .

The addition of the term  $\eta \sum_{j\sigma} (\bar{c}_{j\sigma}^\dagger \bar{c}_{j+1\bar{\sigma}}^\dagger + \text{H.c.})$  to the  $t$ - $J_z$  Hamiltonian has two different effects in the lowest energy subspace  $\mathcal{M}_0$ . The process where the pair of up and down particles created preserves the Néel ordering of the spins can be mapped into the creation of two spinless particles in the effective spinless model. The other possible process creates at least one ferromagnetic link in the parent state and connects  $\mathcal{M}_0$  with the subspace containing one ferromagnetic link,  $N_{\uparrow\uparrow}=1$ , (the lowest spin excitation)  $\mathcal{M}_1$ . This means that the subspaces  $\mathcal{M}$  are no longer invariant under the application of the Hamiltonian. However this second process contributes to second order in  $\eta$  ( $\eta^2/\Delta_s$ ) due to the existence of a spin gap between the ground states of  $\mathcal{M}_0$  and  $\mathcal{M}_1$  (see Appendix I). Therefore to first order in  $\eta$ ,  $\mathcal{M}_0$  is still an invariant subspace and the reduced Hamiltonian is a spinless model with a superconducting term

$$H'_0 = H_0 + \eta \sum_j (b_j^\dagger b_{j+1}^\dagger + b_j b_{j+1}) \quad (19)$$

For arbitrary values of  $J_z$ ,  $t$ , and hole density  $\nu$ ,  $H'_0$  is equivalent (via the traditional Jordan-Wigner transformation) to the spin- $\frac{1}{2}$  XYZ chain Hamiltonian (up to an irrelevant constant)

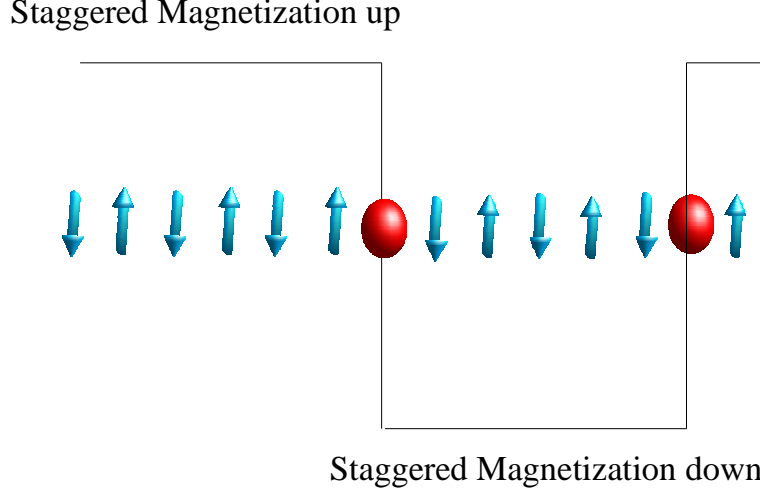
$$H'_0 = \sum_j (\mathcal{J}_x s_j^x s_{j+1}^x + \mathcal{J}_y s_j^y s_{j+1}^y + \mathcal{J}_z (s_j^z s_{j+1}^z + s_j^z)) \quad (20)$$

and  $\mathcal{J}_x = 2(\eta - t)$ ,  $\mathcal{J}_y = -2(\eta + t)$ , and  $\mathcal{J}_z = -\frac{J_z}{4}$ . In the language of our original Hamiltonian  $H_{\text{xxz}}$ , Eq. (11),  $\mathcal{J}_x = 2(\eta + \Delta)$ ,  $\mathcal{J}_y = -2(\eta - \Delta)$ , and  $\mathcal{J}_z = -1$ . From exact solution of this model [9], it is seen that the system is critical only when  $\eta = 0$  while for  $\eta \neq 0$  a gap to all excitations opens.

It is important to note that the ground state of the  $t$ - $J_z$  model,  $|\Psi_0^0\rangle$ , has the same topological long range order as the valence-bond-solid (Haldane state) [10],

i.e., the correlation function [11]  $\langle \Psi_0^0 | S_j^z \exp[i\pi \sum_{k=j+1}^{j+r-1} S_k^z] S_{j+r}^z | \Psi_0^0 \rangle = -\langle \Psi_0^0 | n_j n_{j+r} | \Psi_0^0 \rangle$

has a power law decay as a function of distance  $r$  to a constant value for the  $t$ - $J_z$



**Figure 2.** Each hole carries an anti-phase domain wall in the exact ground state of the  $t$ - $J_z$  model.

point. This means that the superconducting term in  $H_{\text{xxz}}$ , although it is opening a gap (Haldane gap), it is not changing the topological order characterizing the ground state. Therefore, the ground state is in the same Haldane phase for  $0 \leq \eta \leq \Delta$ . In the particle language, the Haldane gap is a superconducting gap. Since each hole ( $S^z = 0$  state in the spin language) is an anti-phase boundary (soliton) for the Néel ordering (see Fig. (2)), the AF correlation function is short ranged for the ground state of the  $t$ - $J_z$  model. As demonstrated above, these solitonic excitations are massless at the  $t$ - $J_z$  point, but become massive (gapped) as soon as the superconducting term is turned on ( $\eta \neq 0$ ). As the superconducting term is derived from the transverse part of the Heisenberg interaction, it will not restore the AF ordering along the  $z$  direction. In this way it is easy to understand why the spin-spin correlation function, of the  $S=1$  AF Heisenberg chain is short ranged:  $\langle \mathbf{S}_j \cdot \mathbf{S}_{j+r} \rangle \sim \exp[-r/\xi]$ .

### 3.2 $S=1$ Integrable Models and Hidden Symmetries

To illustrate further the power of our spin-fermion mapping we now present *exact* solutions of 1d  $S=1$  models that have not been discovered by traditional techniques. These models correspond to the family of bilinear-biquadratic Hamiltonians

$$H_1(\Delta) = \sum_j H_j^z + H_j^{\text{xx}} + \{H_j^z, H_j^{\text{xx}}\} = \sum_j H_j^z + \Delta \sum_{\sigma} \bar{c}_{j\sigma}^{\dagger} \bar{c}_{j+1\sigma} , \quad (21)$$

that can be mapped onto a ( $S = \frac{1}{2}$ )  $t$ - $J_z$  Hamiltonian, whose quantum phase diagram has recently been exactly solved [7].

Another well-studied class of bilinear-biquadratic  $SU(2)$  invariant Hamiltonians is [12]

$$H_2(\Delta) = \sum_j \mathbf{S}_j \cdot \mathbf{S}_{j+1} + \Delta (\mathbf{S}_j \cdot \mathbf{S}_{j+1})^2 , \quad (22)$$

where  $-1 \leq \Delta \leq 1$ . The pure Heisenberg ( $\Delta=0$ ) and Valence Bond Solid models ( $\Delta = \frac{1}{3}$ ) belong to the Haldane gapped phase, that extends over the whole interval of  $\Delta$  except at the boundaries  $\Delta = \pm 1$  which are quantum critical points. The case  $\Delta = -1$  is known to be Bethe ansatz solvable with a unique ground state and gapless. For  $\Delta=1$  we can map  $H_2(1)$  onto the supersymmetric ( $S=\frac{1}{2}$ )  $t$ - $J$  Hamiltonian plus a NN repulsive interaction

$$H_2(1) = - \sum_{j\sigma} \left( \bar{c}_{j\sigma}^\dagger \bar{c}_{j+1\sigma} + \text{H.c.} \right) + 2 \sum_j \mathbf{s}_j \cdot \mathbf{s}_{j+1} + 2 \sum_j (1 - \bar{n}_j + \frac{3}{4} \bar{n}_j \bar{n}_{j+1}) , \quad (23)$$

where  $\mathbf{s}_j$  represents a  $S=\frac{1}{2}$  operator. This model is Bethe-ansatz solvable with a gapless phase [13] and is known as the Lai-Sutherland solution [14].

We now discuss the importance of our generalized Jordan-Wigner transformation in unraveling hidden symmetries of an arbitrary spin Hamiltonian. In Eq. (22), for example, the  $S=1$   $SU(2)$  symmetry is manifest. However, both the  $S=\frac{1}{2}$   $SU(2)$  and global  $U(1)$  gauge symmetries are hidden. On the other hand, in the transformed Hamiltonian, Eq. (23), these two symmetries are manifested explicitly through rotational invariance and charge conservation. The generators of these symmetries are related through the mapping already introduced. To illustrate this, we consider the  $U(1)$  symmetry case. Here the generator of the transformation is  $Q = \sum_j \bar{n}_j$  which maps onto  $Q = \sum_j (S_j^z)^2$  in the spin representation. The total group symmetry of the Hamiltonian is  $SU(3)$ . To see this explicitly, let us rewrite Eq. (23) in a way that is explicitly  $SU(3)$  invariant (up to an irrelevant constant) [16]

$$H_2(1) = \sum_j \mathcal{S}^{\mu\nu}(j) \mathcal{S}_{\nu\mu}(j+1) ,$$

where  $\mathcal{S}^{\mu\nu}(j)$  is a nine component tensor (traceless, i.e.,  $\text{Tr}[\mathcal{S}] = 0$ )

$$\mathcal{S}(j) = \begin{pmatrix} \bar{n}_{j\uparrow} - \frac{1}{3} & \bar{c}_{j\uparrow}^\dagger K_j & \bar{c}_{j\downarrow}^\dagger K_j \\ K_j^\dagger \bar{c}_{j\uparrow} & \bar{n}_{j\downarrow} - \frac{1}{3} & \bar{c}_{j\uparrow}^\dagger \bar{c}_{j\downarrow} \\ K_j^\dagger \bar{c}_{j\downarrow} & \bar{c}_{j\downarrow}^\dagger \bar{c}_{j\uparrow} & -(\bar{n}_{j\uparrow} + \bar{n}_{j\downarrow} - \frac{2}{3}) \end{pmatrix} ,$$

whose eight components ( $\mathcal{S}^{33}(j)$  is linearly dependent) constitute a basis of the  $su(3)$  algebra [16].

#### 4. GENERALIZED TRANSFORMATIONS

A general transformation for arbitrary spin and spatial dimension is the following (see Fig. (3))



**Half-odd integer spin**  $S$  ( $\sigma \in \mathcal{F}_{\frac{1}{2}} = \{-S+1, \dots, S\}$ ):

$$\begin{aligned}
S_j^+ &= \eta_{\bar{S}} \bar{c}_{j\bar{S}+1}^\dagger K_j + \sum_{\substack{\sigma \in \mathcal{F}_{\frac{1}{2}} \\ \sigma \neq S}} \eta_\sigma \bar{c}_{j\sigma+1}^\dagger \bar{c}_{j\sigma} , \\
S_j^- &= \eta_{\bar{S}} K_j^\dagger \bar{c}_{j\bar{S}+1} + \sum_{\substack{\sigma \in \mathcal{F}_{\frac{1}{2}} \\ \sigma \neq S}} \eta_\sigma \bar{c}_{j\sigma}^\dagger \bar{c}_{j\sigma+1} , \\
S_j^z &= -S + \sum_{\sigma \in \mathcal{F}_{\frac{1}{2}}} (S + \sigma) \bar{n}_{j\sigma} , \\
\bar{c}_{j\sigma}^\dagger &= K_j^\dagger L_\sigma^{\frac{1}{2}} (S_j^+)^{\sigma+S} \mathcal{P}_j^{\frac{1}{2}} , \text{ where } \mathcal{P}_j^{\frac{1}{2}} = \prod_{\tau \in \mathcal{F}_{\frac{1}{2}}} \frac{\tau - S_j^z}{\tau + S} , \quad L_\sigma^{\frac{1}{2}} = \prod_{\tau=-S}^{\sigma-1} \eta_\tau^{-1} . \quad (24)
\end{aligned}$$

**Integer spin**  $S$  ( $\sigma \in \mathcal{F}_1 = \{-S, \dots, -1, 1, \dots, S\}$ ):

$$\begin{aligned}
S_j^+ &= \eta_0 (\bar{c}_{j1}^\dagger K_j + K_j^\dagger \bar{c}_{j\bar{1}}) + \sum_{\substack{\sigma \in \mathcal{F}_1 \\ \sigma \neq -1, S}} \eta_\sigma \bar{c}_{j\sigma+1}^\dagger \bar{c}_{j\sigma} , \\
S_j^- &= \eta_0 (K_j^\dagger \bar{c}_{j1} + \bar{c}_{j\bar{1}}^\dagger K_j) + \sum_{\substack{\sigma \in \mathcal{F}_1 \\ \sigma \neq -1, S}} \eta_\sigma \bar{c}_{j\sigma}^\dagger \bar{c}_{j\sigma+1} , \\
S_j^z &= \sum_{\sigma \in \mathcal{F}_1} \sigma \bar{n}_{j\sigma} , \\
\bar{c}_{j\sigma}^\dagger &= K_j^\dagger L_\sigma^1 \begin{cases} (S_j^+)^\sigma \mathcal{P}_j^1 , & \text{if } \sigma > 0 \\ (S_j^-)^\sigma \mathcal{P}_j^1 , & \text{if } \sigma < 0 \end{cases} , \quad (25)
\end{aligned}$$

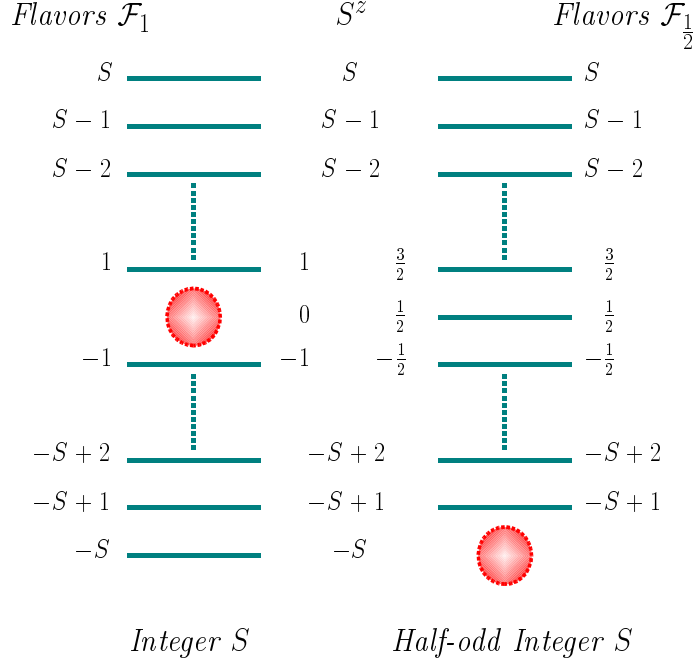
where  $\mathcal{P}_j^1 = \prod_{\tau \in \mathcal{F}_1} \frac{\tau - S_j^z}{\tau}$  ,  $L_\sigma^1 = \prod_{\tau=0}^{|\sigma|-1} \eta_\tau^{-1}$  and  $\eta_\sigma = \sqrt{(S-\sigma)(S+\sigma+1)}$ .

The total number of flavors is  $N_f = 2S$ , and the  $S=\frac{1}{2}$  case simply reduces to the traditional Jordan-Wigner transformation. Since these mappings are exact they preserve the invariant Casimir operator  $\mathbf{S}_j^2 = S(S+1)$ . The generalized constrained fields

$$\bar{c}_{j\sigma}^\dagger = c_{j\sigma}^\dagger \prod_{\tau \in \mathcal{F}_\alpha} (1 - n_{j\tau}) , \quad \bar{c}_{j\sigma} = \prod_{\tau \in \mathcal{F}_\alpha} (1 - n_{j\tau}) c_{j\sigma} \quad (26)$$

form a subalgebra of the generalized Hubbard double graded algebra, where the “unconstrained” operators  $c_{j\sigma}^\dagger, c_{j\sigma}$  satisfy the standard fermion anticommutation relations ( $\alpha = \frac{1}{2}, 1$  depending upon the spin character of the representation). These generalized constrained operators (only single occupancy is allowed) anticommute for different sites

$$\{\bar{c}_{j\sigma}, \bar{c}_{k\sigma'}\} = \{\bar{c}_{j\sigma}^\dagger, \bar{c}_{k\sigma'}^\dagger\} = 0 , \quad \{\bar{c}_{j\sigma}, \bar{c}_{k\sigma'}^\dagger\} = \delta_{jk} \begin{cases} \prod_{\substack{\tau \in \mathcal{F}_\alpha \\ \tau \neq \sigma}} (1 - \bar{n}_{j\tau}) , & \text{if } \sigma = \sigma' , \\ \bar{c}_{j\sigma'}^\dagger \bar{c}_{j\sigma} , & \text{if } \sigma \neq \sigma' , \end{cases} \quad (27)$$



**Figure 3.** Constrained fermion states per site for integer and half-odd integer spin  $S$ . In both cases there are  $2S$  flavors and the corresponding  $2S + 1$  values of  $S^z$  are shown in the middle column. One degree of freedom is assigned to the fermion vacuum (circle) whose relative position depends upon the spin character.

and their number operators satisfy  $\bar{n}_{j\sigma}\bar{n}_{j\sigma'} = \delta_{\sigma\sigma'}\bar{n}_{j\sigma}$ .

The string operators  $K_j$  introduce nonlinear and nonlocal interactions between the constrained fermions. For  $1d$  lattices ( $K_j = K_j^\dagger$ ,  $[K_i, K_j] = 0$ ) they are the so-called kink operators  $K_j = \exp[i\pi \sum_{k < j} \bar{n}_k]$ , while for  $2d$  [15]

$$K_{\mathbf{j}} = \exp[i \sum_{\mathbf{k}} a(\mathbf{k}, \mathbf{j}) \bar{n}_{\mathbf{k}}] , \quad \text{with } \bar{n}_{\mathbf{k}} = \sum_{\sigma \in \mathcal{F}_\alpha} \bar{n}_{\mathbf{k}\sigma} = 1 - \mathcal{P}_{\mathbf{k}}^\alpha . \quad (28)$$

Here,  $a(\mathbf{k}, \mathbf{j})$  is the angle between the spatial vector  $\mathbf{k} - \mathbf{j}$  and a fixed direction on the lattice, and  $a(\mathbf{j}, \mathbf{j})$  is defined to be zero. We comment that the  $1d$  kink operators constitute a particular case of Eq. (28) with  $a(k, j) = \pi$  when  $k < j$  and equals zero otherwise. For  $d > 2$ , the string operators generalize [16] along the lines introduced in Ref. [17].

Why are the vacuum states chosen like in Fig. (3)? Note that from a mathematical viewpoint the vacuum could be identified with any  $S^z$  value. However, our choice is the most symmetric one and therefore the most useful one to establish connections between Hamiltonians which are relevant in different fields of physics. The identification of the vacuum state with  $S^z = 0$  can only be done in the integer case.

There is always the freedom to perform rotations in spin space to get equivalent representations to the one presented above. However, for bilinear isotropic NN Heisenberg (spin  $SU(2)$  rotationally invariant) Hamiltonians in the large- $S$  limit,

there is a fundamental difference between effective integer and half-odd integer spin cases. In the latter case a new local  $U(1)$  gauge symmetry emerges that is explicitly broken in the integer case. For  $1d$  lattices, this is precisely what distinguishes Haldane gap systems [6] from half-odd integer spin chains that are critical.

We mention that other fermionic representations are feasible. In particular, for half-odd integer cases where  $2S + 1 = \sum_{i=0}^{\bar{N}_f} \binom{\bar{N}_f}{i} = 2^{\bar{N}_f}$  (e.g.,  $S = \frac{3}{2}$  with  $\bar{N}_f = 2$ ) a simple transformation in terms of standard “unconstrained” fermions is possible [16]. For these mappings the string operators must be modified to take into account the double occupancy of a site. In this way the Hubbard model can be mapped onto a  $S = \frac{3}{2}$  spin Hamiltonian [16].

## 5. TWO DIMENSIONAL LATTICES AND SPIN-ANYON MAPPING

The generalization of these transformations to higher dimensions gives new exact mappings between spin theories and constrained fermion systems in the presence of gauge fields. To illustrate this we write the  $S=1$  Hamiltonian  $H_2(1)$  in the fermion representation for  $d=2$

$$H_2(1) = - \sum_{\mathbf{j}, \nu} \left( \bar{c}_{\mathbf{j}+\mathbf{e}_\nu, \sigma}^\dagger e^{iA_\nu(\mathbf{j})} \bar{c}_{\mathbf{j}, \sigma} + \text{H.c.} \right) + 2 \sum_{\mathbf{j}, \nu} \mathbf{s}_{\mathbf{j}} \cdot \mathbf{s}_{\mathbf{j}+\mathbf{e}_\nu} + \sum_{\mathbf{j}, \nu} \left( 2 - (\bar{n}_{\mathbf{j}} + \bar{n}_{\mathbf{j}+\mathbf{e}_\nu}) + \frac{3}{2} \bar{n}_{\mathbf{j}} \bar{n}_{\mathbf{j}+\mathbf{e}_\nu} \right), \quad (29)$$

and

$$A_\nu(\mathbf{j}) = \sum_{\mathbf{k}} [a(\mathbf{k}, \mathbf{j}) - a(\mathbf{k}, \mathbf{j} + \mathbf{e}_\nu)] \bar{n}_{\mathbf{k}}, \quad (30)$$

where  $\mathbf{e}_\nu$  ( $\nu = 1, 2$ ) are basis vectors of the Bravais lattice connecting NN and  $\mathbf{j}$ 's represent sites of the corresponding  $2d$  lattice. We note that the field  $A_\nu(\mathbf{j})$  is associated with the change in particle statistics. It is well-known [15,3] that the same transmutation of particle statistics can be achieved via a path-integral formulation for  $H_2(1)$  where an Abelian lattice Chern-Simons term is included. In this formulation a constraint (Gauss's law) requiring that the gauge flux through a plaquette  $\mathbf{j}$  be proportional to the total fermion density on the site,  $\bar{n}_{\mathbf{j}}$ , is enforced. This suggests that our spin-fermion mapping can be generalized to an spin-anyon transformation with a hard-core condition for the anyon fields [16]. In fact, one can formally take our generalized Jordan-Wigner transformation and replace the string operators  $K_{\mathbf{j}}$  by the statistical operators  $K_{\mathbf{j}}(\theta) = \exp[i\theta \sum_{\mathbf{k}} a(\mathbf{k}, \mathbf{j}) \bar{n}_{\mathbf{k}}]$  with  $0 \leq \theta \leq 1$ . With this choice, the  $\bar{c}$  operators satisfy equal-time anyon commutation relations [ $\theta = 1(0)$  corresponds to constrained fermions(bosons)] [16]. Similar ideas apply for  $1d$  lattices.

One immediately sees the relevance of these transformations for the theories of magnetism and high-temperature superconductivity: A class of  $S=1$  Hamiltonians that can be mapped onto a lattice-gauge (Chern-Simons)  $S = \frac{1}{2} t$ - $J$  theory and vice versa; for example, a  $S = \frac{1}{2} t$ - $J$  model,

$$H_{t-J} = -t \sum_{\mathbf{j}, \nu} \left( \bar{c}_{\mathbf{j}, \sigma}^\dagger \bar{c}_{\mathbf{j}+\mathbf{e}_\nu, \sigma} + \text{H.c.} \right) + J \sum_{\mathbf{j}, \nu} \mathbf{s}_{\mathbf{j}} \cdot \mathbf{s}_{\mathbf{j}+\mathbf{e}_\nu} - \mu \sum_{\mathbf{j}} \bar{n}_{\mathbf{j}}, \quad (31)$$

can be exactly mapped onto a lattice-gauge bilinear-biquadratic  $S=1$  theory

$$H_{t-J} = \frac{J}{8} \sum_{\mathbf{j}, \nu} \left( H_{\mathbf{j}\nu}^z - \frac{4t}{J} S_{\mathbf{j}}^+ e^{iA_{\nu}(\mathbf{j})} S_{\mathbf{j}+\mathbf{e}_{\nu}}^- - \frac{4t}{J} \left\{ H_{\mathbf{j}\nu}^z, S_{\mathbf{j}}^+ e^{iA_{\nu}(\mathbf{j})} S_{\mathbf{j}+\mathbf{e}_{\nu}}^- \right\} \right. \\ \left. + (S_{\mathbf{j}}^+ S_{\mathbf{j}+\mathbf{e}_{\nu}}^-)^2 + \text{H.c.} \right) - \mu \sum_{\mathbf{j}} (S_{\mathbf{j}}^z)^2. \quad (32)$$

By means of a semiclassical approximation it has been shown [18] that the ground state of  $H_2(1)$  is on the boundary between AF ( $\Delta < 1$ ) and orthogonal nematic (non-uniform,  $\Delta > 1$ ) phases [18,12]. These two states are the result of the competition between the quadratic and quartic spin-exchange interactions in  $H_2(\Delta)$ . In terms of the equivalent  $t$ - $J$  gauge theory this translates into a competition between antiferromagnetism and delocalization. Qualitatively, the string-path of the particle moving in an AF background gives rise to a linear confining potential since the number of frustrated magnetic links is proportional to the length of the path. This observation suggests that the inhomogeneous phases observed in the “striped” high- $T_c$  compounds can be driven by the competition between magnetism and delocalization.

## 6. SUMMARY

We introduced a general spin-particle mapping for arbitrary spin  $S$  and spatial dimension that naturally generalizes the Jordan-Wigner transformation for  $S = \frac{1}{2}$ . Indeed our transformations define exact connections between localized quantum spins  $S$  on a lattice and quantum lattice gases of itinerant particles with “effective” spin  $s = S - \frac{1}{2}$ . Mathematically, we established a one-to-one mapping of elements of a Lie algebra onto elements of a fermionic algebra with a hard-core constraint. Several generalizations, like a spin-anyon mapping, and important consequences result from these transformations [16]. Incidentally, we note that there are extremely powerful numerical techniques (cluster algorithms [19]) to study quantum spin systems, and our mapping allows one to extend these methods to study the equivalent fermionic problems. Generalizations of these spin-particle transformations to  $SU(N)$  algebras exist [16]. As it has been illustrated in a previous Section this type of mappings is very useful to find particle models having internal symmetries which simplify the resolution of those problems. In this way it is possible to find exact solutions in any dimension having more than one spontaneously broken symmetry. For instance, we have found exact solutions in any spatial dimension for  $S = \frac{1}{2}$  hard-core bosons which are magnetic Bose-Einstein condensates. On the other hand, the existence of a Quantum Link Model connecting Lattice-Gauge theories to spin theories in addition to the transformations introduced in this paper, opens the possibility of formal connections between gauge theories of high-energy physics and strongly correlated problems of condensed matter. Finally, it is important to mention that there are other possible spin-particle mappings such as connections between spins and canonical fermions. However, the latter transformations are only feasible for some irreducible spin representations.

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## APPENDIX I

In this Appendix we prove that the 1d  $t$ - $J_z$  model has a gap to spin excitations. We follow the notation of Section (3.1).

The spin gap is defined as

$$\Delta_s = \langle \Psi_0^1 | H_0^1 | \Psi_0^1 \rangle - \langle \Psi_0^0 | H_0^0 | \Psi_0^0 \rangle \geq 0, \quad (A1)$$

where  $H_0^0$  is the Hamiltonian restricted to the  $\mathcal{M}_0$  subspace, while  $H_0^1$  is the one restricted to  $\mathcal{M}_1$  ( $\mathcal{M}_1$  is the subspace generated from a Néel ordered parent state with  $N_{\uparrow\uparrow} = 1$ )

$$\mathcal{M}_0 : \underbrace{|\uparrow\downarrow\uparrow\downarrow \cdots \uparrow^\nu \downarrow\uparrow\downarrow\uparrow \cdots \circ \circ \circ \circ \cdots\rangle}_{L-M} \text{ and } \mathcal{M}_1 : \underbrace{|\uparrow\downarrow\uparrow\downarrow \cdots \uparrow^\nu \uparrow\downarrow\uparrow\downarrow \cdots \circ \circ \circ \circ \cdots\rangle}_{L-M} \text{ and } \underbrace{\quad}_{M}.$$

$|\Psi_0^0\rangle$  and  $|\Psi_0^1\rangle$  are the normalized ground states ( $\langle \Psi_0^i | \Psi_0^i \rangle = 1$ ) of  $H_0^0$  and  $H_0^1$ , respectively. From the Rayleigh-Ritz variational principle

$$\Delta_s \geq \langle \Psi_0^1 | H_0^1 | \Psi_0^1 \rangle - \langle \Psi_0^1 | H_0^0 | \Psi_0^1 \rangle = \langle \Psi_0^1 | H_0^1 - H_0^0 | \Psi_0^1 \rangle. \quad (A2)$$

Since  $H_0^1 - H_0^0 = \frac{J_z}{2} n_\nu n_{\nu+1}$  we have to show that the correlation function  $\tilde{\Delta} = \langle \Psi_0^1 | n_\nu n_{\nu+1} | \Psi_0^1 \rangle$  is finite in order to demonstrate that there is a finite spin gap  $\Delta_s$ .

The wave function  $|\Psi_0^1\rangle$  can be written as

$$|\Psi_0^1\rangle = \sum_{i_1 < i_2 < \cdots < i_{L-M}} a_I b_{i_1}^\dagger b_{i_2}^\dagger \cdots b_{i_\nu}^\dagger b_{i_{\nu+1}}^\dagger \cdots b_{i_{L-M}}^\dagger |0\rangle \quad (A3)$$

in the basis  $\{b_{i_1}^\dagger b_{i_2}^\dagger \cdots b_{i_\nu}^\dagger b_{i_{\nu+1}}^\dagger \cdots b_{i_{L-M}}^\dagger |0\rangle\}$  with  $I = (i_1, i_2, \cdots, i_{L-M})$  and  $i_\mu \in [1, L]$ . [ $I$  represents a generic particle configuration. Note that  $i_\nu = \nu$  only for the parent state particle configuration.] If one assumes that  $\tilde{\Delta} = 0(\mathcal{O}(1/L))$ , then  $|\Psi_0^1\rangle$  has zero (or infinitesimal) projection on the subspace of states where  $i_\nu + 1 = i_{\nu+1}$ . This means that there must be at least one empty site between particles  $\nu$  and  $\nu + 1$  in all the terms of  $|\Psi_0^1\rangle$ , i.e.,  $i_\nu + \alpha = i_{\nu+1}$  with  $\alpha > 1$  (the rest is a set of zero measure). Different values of  $\alpha$  define different subspaces of the Hilbert space:  $\{b_{i_1}^\dagger b_{i_2}^\dagger \cdots b_{i_\nu}^\dagger b_{i_{\nu+\alpha}}^\dagger \cdots b_{i_{L-M}}^\dagger |0\rangle\}$ . For finite concentration of particles there is a minimum value  $\alpha_m$  for which the wave function  $|\Psi_0^1\rangle$  has a finite projection onto the corresponding subspace. We call the projected wave function  $|\Psi_p\rangle = P_{\alpha_m} |\Psi_0^1\rangle$  with

$$P_\alpha = \sum_{\substack{i_1 < i_2 < \cdots < i_{L-M} \\ i_\nu + \alpha = i_{\nu+1}}} n_{i_1} n_{i_2} \cdots n_{i_{L-M}} \quad (A4)$$

with  $n_{i_\mu} = b_{i_\mu}^\dagger b_{i_\mu}$ . We define now the state

$$\sqrt{\langle \Psi_p | \Psi_p \rangle} |\Psi_1\rangle = \hat{O}_{\nu, \alpha_m} |\Psi_0^1\rangle = \sum_{\substack{i_1 < i_2 < \dots < i_{L-M}, \\ i_\nu + \alpha_m = i_{\nu+1}}} a_I b_{i_1}^\dagger b_{i_2}^\dagger \dots b_{i_{\nu+1}}^\dagger b_{i_{\nu+1}}^\dagger b_{i_{L-M}}^\dagger |0\rangle \quad (A5)$$

with

$$\hat{O}_{\nu, \alpha} = \sum_{\substack{i_1 < i_2 < \dots < i_{L-M}, \\ i_\nu + \alpha = i_{\nu+1}}} n_{i_1} n_{i_2} \dots b_{i_{\nu+1}}^\dagger b_{i_\nu} n_{i_{\nu+1}} \dots n_{i_{L-M}} \quad (A6)$$

$|\Psi_1\rangle$  is a normalized state which is orthogonal to  $|\Psi_0^1\rangle$  because it belongs to the subspace  $\alpha_m - 1$ .

One can show that  $H_0^1$  connects the states  $|\Psi_0^1\rangle$  and  $|\Psi_1\rangle$  with the result

$$\langle \Psi_1 | H_0^1 | \Psi_0^1 \rangle = -2t \langle \Psi_p | \Psi_p \rangle + \mathcal{O}(1/L) > 0 \quad (A7)$$

One can also compute  $\langle \Psi_1 | H_0^1 | \Psi_1 \rangle$

$$\langle \Psi_1 | H_0^1 | \Psi_1 \rangle = \frac{\langle \Psi_0^1 | \hat{O}_{\nu, \alpha_m}^\dagger H_0^1 \hat{O}_{\nu, \alpha_m} | \Psi_0^1 \rangle}{\langle \Psi_p | \Psi_p \rangle}. \quad (A8)$$

Then,

$$\langle \Psi_1 | H_0^1 | \Psi_1 \rangle = \frac{\langle \Psi_0^1 | \hat{O}_{\nu, \alpha_m}^\dagger \hat{O}_{\nu, \alpha_m} H_0^1 | \Psi_0^1 \rangle}{\langle \Psi_p | \Psi_p \rangle} + \frac{\langle \Psi_0^1 | \hat{O}_{\nu, \alpha_m}^\dagger [H_0^1, \hat{O}_{\nu, \alpha_m}] | \Psi_0^1 \rangle}{\langle \Psi_p | \Psi_p \rangle}, \quad (A9)$$

and, therefore

$$\langle \Psi_1 | H_0^1 | \Psi_1 \rangle = E_0^1 + \frac{\langle \Psi_0^1 | \hat{O}_{\nu, \alpha_m}^\dagger [H_0^1, \hat{O}_{\nu, \alpha_m}] | \Psi_0^1 \rangle}{\langle \Psi_p | \Psi_p \rangle}, \quad (A10)$$

where we have used that  $|\Psi_0^1\rangle$  is the ground state of  $H_0^1$ ,  $H_0^1 |\Psi_0^1\rangle = E_0^1 |\Psi_0^1\rangle$  and  $\langle \Psi_0^1 | \hat{O}_{\nu, \alpha_m}^\dagger \hat{O}_{\nu, \alpha_m} | \Psi_0^1 \rangle = \langle \Psi_p | \Psi_p \rangle$ . The second term in Eq. (A10) is bounded ( $\mathcal{O}(1)$ ) since  $[H_0^1, \hat{O}_{\nu, \alpha}]$  is an operator which only affects the particles  $i_\nu$  and  $i_{\nu+1}$ . We can then build a state which is a linear combination of  $|\Psi_0^1\rangle$  and  $|\Psi_1\rangle$  having an energy lower than  $E_0^1$  because  $H_0^1$  connects the two states. But this contradicts the initial hypothesis which stated that  $|\Psi_0^1\rangle$  was the ground state of  $H_0^1$ . Therefore,  $\tilde{\Delta}$  is finite and there is a spin gap.

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$\mathbb{1}$  is the  $(2S + 1) \times (2S + 1)$  unit matrix and  $S^\mu$  is a spin- $S$  operator. Thus  $S_j^\mu$  admits a matrix representation of dimension  $(2S + 1)^L \times (2S + 1)^L$ .

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